

Phong Normalization Factor derivation

I'll do pure-specular only (i.e. $C_d = 0$, $C_s = 1$), the mixed case is easy from there. Also, we're only interested in the maximum of reflected energy, which in the Phong model occurs when L and N are parallel to each other, which makes $R = N$ too (in all other cases, R is "on the other side" of N relative to V , hence the angle between R and V can never be smaller than the angle between R and N). Anyway, this means that $R \cdot V = N \cdot V$, which is a value we already know, namely $\cos \theta$.

Moving on, the integral we now need to calculate is

$$\int_{\Omega} (\cos \theta)^n d\omega \quad (1)$$

with Ω being the upper hemisphere; integrating in spherical coordinates, this is

$$\int_0^{2\pi} \int_0^{\pi/2} (\cos \theta)^n \sin \theta d\theta d\phi = 2\pi \int_0^{\pi/2} (\cos \theta)^n \sin \theta d\theta =: 2\pi I_n \quad (2)$$

and using integration by parts with $f = (\cos \theta)^n$, $g' = \sin \theta$ on I_n we get

$$\begin{aligned} I_n &= [(\cos \theta)^n (-\cos \theta)]_0^{\pi/2} - \int_0^{\pi/2} n(\cos \theta)^{n-1} (-\sin \theta) (-\cos \theta) d\theta \\ &= [-(\cos \theta)^{n+1}]_0^{\pi/2} - n \int_0^{\pi/2} (\cos \theta)^n \sin \theta d\theta \\ &= (-0 + 1) - n I_n \end{aligned}$$

so $(n+1)I_n = 1$ which means that $I_n = \frac{1}{n+1}$. Plugging this into (2) tells us that (1) equals $\frac{2\pi}{n+1}$, so the normalization factor if we want it to integrate to 1 is the reciprocal, which is $\frac{n+1}{2\pi}$.

Why $\frac{n+1}{2}$ and not $n+2$? Because this is the derivation for the original Phong formulation, where the $R \cdot V$ term is not multiplied by $\cos \theta$. If you write that version of the Phong model as a BRDF, you end up with a $\cos \theta$ in the numerator to cancel out the $\cos \theta$ factor in the reflection equation. This numerator is complete nonsense physically, so the modern formulation of the Phong model removes it. Then the integral becomes

$$\int_{\Omega} (R \cdot V) \cos \theta d\omega \stackrel{L \equiv N}{=} \int_{\Omega} (\cos \theta)^{n+1} d\omega$$

and our normalization factor computation cranks out $\frac{n+2}{2}$, as expected.

Blinn-Phong normalization factor

I'll again limit myself to the specular term and again assume that the maximum reflected energy occurs with $L = N$ (I have no proof for the latter though, but I do have some experimental evidence. If I find a nice proof later, I'll update this document accordingly. Anyway, with $L = N$, things get a lot simpler than the general case because L , N , V , and H all lie in the same plane and we can work exclusively with angles. Particularly, the angle θ_h between H and N is exactly half of the angle θ between V and N , and the integral we need to evaluate boils down to

$$\int_{\Omega} (\cos \theta_h)^n \cos \theta \, d\omega = \int_{\Omega} (\cos \theta/2)^n \cos \theta \, d\omega$$

(I'll only do the BRDF version with the extra factor of $\cos \theta$ here). Again integrating in spherical coordinates, we get

$$\int_0^{2\pi} \int_0^{\pi/2} (\cos \theta/2)^n \cos \theta \sin \theta \, d\theta d\phi = 2\pi \int_0^{\pi/2} (\cos \theta/2)^n \cos \theta \sin \theta \, d\theta \quad (3)$$

and using the half-angle formula $\cos(\theta/2) = \sqrt{\frac{1+\cos \theta}{2}}$ and the substitution $t = \cos \theta$ (which gives $dt = -\sin \theta \, d\theta$) we get

$$(3) = -2\pi \int_1^0 \left(\sqrt{\frac{1+t}{2}} \right)^n t \, dt = 2\pi \int_0^1 \left(\frac{1+t}{2} \right)^{n/2} t \, dt$$

which can be evaluated using integration by parts, this time using $f = t$ and $g' = ((1+t)/2)^{n/2}$. This yields:

$$\begin{aligned} & 2\pi \left(\left[\frac{4}{n+2} t \left(\frac{1+t}{2} \right)^{(n+2)/2} \right]_{t=0}^1 - \frac{4}{n+2} \int_0^1 \left(\frac{1+t}{2} \right)^{(n+2)/2} dt \right) \\ &= \frac{8\pi}{n+2} \left(\left[t \left(\frac{1+t}{2} \right)^{(n+2)/2} \right]_{t=0}^1 - \frac{4}{n+4} \left[\left(\frac{1+t}{2} \right)^{(n+4)/2} \right]_{t=0}^1 \right) \\ &= \frac{8\pi}{n+2} \left[\cos(\theta) \cos(\theta/2)^{n+2} - \frac{4}{n+4} \cos(\theta/2)^{n+4} \right]_{\theta=\pi/2}^0 \\ &= \frac{8\pi [\cos(\theta/2)^{n+2} (4 \cos(\theta/2)^2 - (n+4) \cos(\theta))]_{\theta=0}^{\pi/2}}{(n+2)(n+4)} \\ &= \frac{8\pi [\cos(\theta/2)^{n+2} (2(1+\cos(\theta)) - (n+4) \cos(\theta))]_{\theta=0}^{\pi/2}}{(n+2)(n+4)} \\ &= \frac{8\pi [\cos(\theta/2)^{n+2} (2 - (n+2) \cos(\theta))]_{\theta=0}^{\pi/2}}{(n+2)(n+4)} \\ &= \frac{8\pi (2^{1-(n+2)/2} - (2 - (n+2)))}{(n+2)(n+4)} \\ &= \frac{8\pi (2^{-n/2} + n)}{(n+2)(n+4)} \end{aligned}$$

which makes the Blinn-Phong normalization factor $\frac{(n+2)(n+4)}{8\pi(2^{-n/2}+n)}$, not $\frac{n+8}{8\pi}$.